

# Localization and homological stability of configuration spaces

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December 17, 2012

## Abstract

In [Chu12], Church used representation stability to prove that the space of configurations of distinct unordered points in a closed manifold exhibit rational homological stability. A second proof was also given by Randal-Williams in [RW11] using transfer maps. We give a third proof of this fact using localization and rational homotopy theory. This gives new insight into the role that the rationals play in homological stability. Our methods also yield new information about stability for torsion in the homology of configuration spaces of points in a closed manifold.

## 1 Introduction

Throughout this paper,  $M$  will be a smooth connected  $n$ -manifold with  $n \geq 2$ . Let  $C_k(M)$  denote the configuration space of finite subsets of  $M$  of cardinality  $k$ . That is,  $C_k(M) = (M^k - \Delta_{fat})/\Sigma_k$  where  $\Delta_{fat}$  is the fat diagonal and  $\Sigma_k$  is the symmetric group. When  $M$  is the interior of a manifold with non-empty boundary, McDuff in [McD75] defined a stabilization map:

$$t_k : C_k(M) \longrightarrow C_{k+1}(M)$$

involving “bringing a point in from infinity.” McDuff proved that there is a number  $N_k$  depending only on  $k$  and  $M$  such that  $t_k$  induces an isomorphism on groups  $H_i(\cdot; \mathbb{Z})$  for  $i \leq N_k$  and  $\lim_{k \rightarrow \infty} N_k = \infty$  (Theorem 1.2 of [McD75]). Later Segal showed that one can take  $N_k$  to be  $k/2$  (Proposition A.1 of [Seg79]).

The question of homological stability for configuration spaces of particles in closed manifolds was not addressed for over 30 years until the work of Church (Corollary 3 of [Chu12]) and Randal-Williams (Theorem C of [RW11]). There are two main difficulties in studying configuration spaces of particles in closed manifolds. There is no natural map  $C_k(M) \rightarrow C_{k+1}(M)$  and the integral homology of the spaces  $C_k(M)$  do not stabilize (Theorem 2 of [FVB62]). Nevertheless, Church using representation stability and Randal-Williams using transfer maps were able to prove that the rational homology of the spaces  $C_k(M)$  do stabilize.

We give an alternative perspective on this phenomena using localization and construct a zig-zig of maps of spaces between  $C_k(M)$  and  $C_{k+1}(M)$  which induces isomorphism in rational homology in a stable range. We hope that this approach might also be useful for proving rational homological stability theorems in situations where there are no natural stabilization maps or even transfer maps.

We also give conditions for  $C_k(M)$  and  $C_j(M)$  to have isomorphic  $p$ -torsion. Until this point, the only theorems regarding homological stability for torsion are due to Bökigheimer, Cohen and Taylor in [BCT89] (see also [RW11]). Their work yields an explicit calculation of the  $\mathbb{F}_p$ -homology of configuration spaces in the stable range (when  $p=2$  or the manifold is odd dimensional). From these explicit calculations, one can observe that the homology often stabilizes with  $\mathbb{F}_p$  coefficients. In contrast, our results concern the entire  $p$ -torsion subgroups of the integral homology of configuration spaces (see Theorem 4.4). Interestingly, there are situations where the homology with  $\mathbb{F}_p$  coefficients stabilizes but the  $p$ -torsion does not (Theorem 2 of [FVB62]).

**Acknowledgments** We would like to thank Oscar Randal-Williams and Tom Church for several helpful conversations.

## 2 Scanning

Although McDuff did not address the question of homological stability for configuration spaces of points in a closed manifold, she did prove the following theorem (Theorem 1.1 of [McD75]).

**Theorem 2.1** *Let  $M$  be a closed manifold and let  $\dot{T}M \rightarrow M$  denote the fiberwise one point compactification of its tangent bundle. There is a map  $s :$*

$C_k(M) \longrightarrow \Gamma_k(\dot{T}M)$  which induces homology isomorphisms through the same range that  $t_k : C_k(M - pt) \longrightarrow C_{k+1}(M - pt)$  is a homology isomorphism.

Here  $\Gamma_k(\dot{T}M)$  is the space of degree  $k$  sections of  $\dot{T}M \longrightarrow M$ . See [McD75] for the definition of the degree of a section. The map  $s$  is called the scanning map. Combining this theorem with Segal's explicit homological stability range from [Seg79], one gets the following corollary.

**Corollary 2.2** *The map  $s : C_k(M) \longrightarrow \Gamma_k(\dot{T}M)$  induces an isomorphism on  $H_i(\cdot, \mathbb{Z})$  for  $i \leq k/2$ .*

### 3 Rational homotopy

By Corollary 2.2, to prove rational homological stability for the spaces  $C_k(M)$ , it suffices to prove that the spaces  $\Gamma_k(\dot{T}M)$  are rationally homotopic. While this is not always the case, we will prove that  $\Gamma_k(\dot{T}M)$  is rationally homotopic to  $\Gamma_j(\dot{T}M)$  if  $n$  is odd or  $k$  and  $j$  are both not equal to half the Euler characteristic of  $M$ .

Let  $S_{(0)}^n$  denote the rational  $n$ -sphere and let  $\dot{T}M_{(0)} \longrightarrow M$  denote the fiberwise rational localization of the bundle  $\dot{T}M \longrightarrow M$  [Sul74]. The fiberwise localization map  $l : \dot{T}M \longrightarrow \dot{T}M_{(0)}$  induces a localization map between the spaces of sections and hence isomorphisms  $H_*(\Gamma_k(\dot{T}M); \mathbb{Q}) \longrightarrow H_*(\Gamma_k(\dot{T}M_{(0)}); \mathbb{Q})$  (Theorem 5.3 of [Mø187]). While the connected components of  $\Gamma(\dot{T}M)$  are the spaces  $\Gamma_k(\dot{T}M)$  for  $k \in \mathbb{Z}$ , the connected components of  $\Gamma_k(\dot{T}M_{(0)})$  are the spaces  $\Gamma_k(\dot{T}M_{(0)})$  for  $k \in \mathbb{Q}$ . The topology of these spaces depends heavily on whether or not  $n$  is odd. First we will address the case of  $n$  odd.

**Proposition 3.1** *If  $n$  is odd, the connected components of  $\Gamma(\dot{T}M_{(0)})$  are all homotopic.*

**Proof** Since  $S_{(0)}^n \simeq K(\mathbb{Q}, n)$ , it is an infinite loop space and hence an  $H$ -space. Thus,  $\Gamma(\dot{T}M_{(0)})$  is also an  $H$ -space and so its components are all homotopic.

Before we discuss the case of even dimensional manifolds, we will discuss the special case that the tangent bundle is trivial. In this case, we have a natural homotopy equivalence between  $\Gamma_k(\dot{T}M_{(0)})$  and  $Map_k(M, S_{(0)}^n)$ , the space of degree  $k$  maps from  $M$  to  $S_{(0)}^n$ .

**Lemma 3.2** *For  $k$  and  $j$  non-zero, there is a homotopy equivalence between  $\text{Map}_k(M, S_{(0)}^n)$  and  $\text{Map}_j(M, S_{(0)}^n)$ .*

**Proof** Let  $f : S_{(0)}^n \rightarrow S_{(0)}^n$  be a degree  $j/k$  map and let  $g : S_{(0)}^n \rightarrow S_{(0)}^n$  be a degree  $k/j$  map. Since  $f \circ g \simeq g \circ f \simeq \text{id}$ , composition with  $f$  gives a homotopy equivalence between  $\text{Map}_k(M, S_{(0)}^n)$  and  $\text{Map}_j(M, S_{(0)}^n)$  with homotopy inverse given by composition with  $g$ .

We will now show that the bundle  $\dot{T}M_{(0)} \rightarrow M$  is a trivial bundle if  $M$  is orientable.

**Lemma 3.3** *Let  $E \rightarrow M$  be an oriented  $S_{(0)}^n$ -bundle. Then  $E$  is bundle isomorphic to the trivial bundle.*

**Proof** Since  $E$  is oriented, it is classified by a map to  $B\text{Map}_1(S_{(0)}^n, S_{(0)}^n)$ . Using a result of Thom from [Tho56], Møller and Raussen ([MR85] Example 2.5) observed that:

$$\text{Map}_d(S_{(0)}^n, S_{(0)}^n) \simeq \begin{cases} S_{(0)}^n \times S_{(0)}^{n-1} & \text{if } n \text{ is even and } d = 0 \\ S_{(0)}^{2n-1} & \text{if } n \text{ is even and } d \neq 0 \\ S_{(0)}^n & \text{if } n \text{ is odd.} \end{cases}$$

Since  $B\text{Map}_1(S_{(0)}^n, S_{(0)}^n)$  is at least  $n+1$ -connected and  $M$  is  $n$  dimensional, the classifying map of  $E$  is null homotopic.

If  $M$  is orientable, the bundle  $\pi : \dot{T}M \rightarrow M$  is also orientable. It is not true that the bundle isomorphism necessarily preserves the zero section and hence the homotopy equivalence  $\Gamma(\dot{T}M) \simeq \text{Map}(M, S^n)$  might not preserve degree. Note that if  $M$  is orientable, the degree of a section of  $\dot{T}M \rightarrow M$  is the algebraic intersection number of that section with the zero section.

**Proposition 3.4** *If  $n$  is even and  $M$  is orientable, there is a homotopy equivalence  $\Gamma_k(\dot{T}M_{(0)}) \simeq \text{Map}_{k-\chi(M)/2}(M, S_{(0)}^n)$ .*

**Proof** By Lemma 3.3, there is some number  $L$  such that  $\Gamma_k(\dot{T}M_{(0)}) \simeq \text{Map}_{k+L}(M, S_{(0)}^n)$ . This number is the degree of the image of the infinity section of  $\dot{T}M$  in  $\text{Map}(M, S_{(0)}^n)$ . For  $d \neq 0$  and  $n$  even,  $\text{Map}_d(S_{(0)}^n, S_{(0)}^n)$  is  $n+1$ -connected. Thus, if  $f, g : \dot{T}M_{(0)} \rightarrow \dot{T}M_{(0)}$  are two bundle maps which

induce degree  $d$  maps on each fiber, then  $f$  and  $g$  are homotopic. A map of degree  $-1$  from  $S_{(0)}^n$  to  $S_{(0)}^n$  induces a bundle automorphism of the trivial  $S_{(0)}^n$ -bundle over  $M$  and hence a bundle automorphism of  $\dot{T}M_{(0)}$ . Call this bundle map  $f$  and note that  $f$  maps the infinity section into  $\Gamma_{-2L}(\dot{T}M_{(0)})$ .

Pick a metric on  $M$  and let  $g : \dot{T}M \rightarrow \dot{T}M$  be the bundle map which is the function  $v \rightarrow v/|v|^2$  on each fiber. The map  $g$  sends the infinity section to the zero section. The zero section is an element of  $\Gamma_{\chi(M)}(\dot{T}M)$  because the algebraic intersection number of the zero section with itself is the Euler characteristic. Note that  $g$  induces a degree  $-1$  map on each fiber. Extend  $g$  to a bundle automorphism of  $\dot{T}M_{(0)}$ . Since bundle maps of  $\dot{T}M_{(0)}$  are determined up to homotopy by their degree on each fiber, we conclude that  $f \simeq g$  and  $-2L = \chi(M)$ .

We now address the case of non-orientable manifolds.

**Proposition 3.5** *If  $n$  is even and  $M$  is not orientable, there is a homotopy equivalence  $\Gamma_k(\dot{T}M_{(0)}) \simeq \Gamma_j(\dot{T}M_{(0)})$  for all  $k$  and  $j$  not equal to  $\chi(M)/2$ .*

**Proof** For every non-zero  $d \in \mathbb{Q}$ , one can construct a bundle automorphism  $f_d : \dot{T}M_{(0)} \rightarrow \dot{T}M_{(0)}$  which induces a map of degree  $d$  on each fiber. This follows from obstruction theory since the relevant obstructions lie in  $H^k(M; \pi_{k-1}(\text{Map}_d(S_{(0)}^n, S_{(0)}^n))) = 0$ . For the same reason as in the oriented case, we have a unique up to homotopy bundle map of a given degree. Thus,  $f_d \circ f_{1/d}$  is homotopic to the identity.

The bundle maps  $f_d$  induce maps  $\Gamma_k(\dot{T}M_{(0)}) \rightarrow \Gamma_q(\dot{T}S_{(0)}^n)$  for some number  $q \in \mathbb{Q}$ . Our goal is to show that  $q = dk + (1 - d)\chi(M)/2$ . If we could establish this, then the bundle maps would induce homotopy equivalences between every component of  $\Gamma(\dot{T}M_{(0)})$  except for the degree  $\chi(M)/2$  component. Let  $\tilde{M}$  denote the orientation double cover of  $M$ . Since the tangent bundle of  $\tilde{M}$  is the pull back of the tangent bundle of  $M$ , we can lift degree  $k$  sections of  $\dot{T}M_{(0)}$  to degree  $2k$  sections of  $\dot{T}\tilde{M}_{(0)}$  and bundle maps  $f_d$  to bundle maps  $\tilde{f}_d : \dot{T}\tilde{M}_{(0)} \rightarrow \dot{T}\tilde{M}_{(0)}$  which also induce degree  $d$  maps on each fiber. It follows from Proposition 3.4 that composition with  $\tilde{f}_d$  induces a map  $\Gamma_{2k}(\dot{T}\tilde{M}_{(0)}) \rightarrow \Gamma_{2k+(1-d)\chi(\tilde{M})/2}(\dot{T}\tilde{M}_{(0)})$ . Since  $\chi(\tilde{M})/2 = \chi(M)$ , composition with  $f_d$  gives a map between  $\Gamma(\dot{T}M_{(0)})_k$  and  $\Gamma_{dk+(1-d)\chi(M)/2}(\dot{T}M_{(0)})$ . Since these maps are homotopy equivalences,  $\Gamma_k(\dot{T}M_{(0)}) \simeq \Gamma_j(\dot{T}M_{(0)})$  for all  $k$  and  $j$  not equal to  $\chi(M)/2$ .

Combining Proposition 3.1, Lemma 3.2, Proposition 3.4 and Proposition 3.5, we get the following corollary.

**Theorem 3.6** *The rational homology of  $\Gamma_k(\dot{T}M)$  is isomorphic to the rational homology of  $\Gamma_j(\dot{T}M)$  unless  $n$  is even and  $k$  or  $j$  is  $\chi(M)/2$ .*

Combining Corollary 2.2 and Theorem 3.6, we deduce homological stability for configuration spaces of points in a closed manifold.

**Corollary 3.7** *The homology groups  $H_i(C_k(M); \mathbb{Q})$  are equal to those of  $H_i(C_j(M); \mathbb{Q})$  if  $i \leq \min(k/2, j/2)$  and  $k, j \neq \chi(M)/2$ . Moreover, an isomorphism is given by traversing the following diagram:*

$$\begin{array}{ccccc} C_k(M) & \xrightarrow{s} & \Gamma_k(\dot{T}M) & \xrightarrow{l} & \Gamma_k(\dot{T}M_{(0)}) \\ & & & & \downarrow \simeq \\ C_j(M) & \xrightarrow{s} & \Gamma_j(\dot{T}M) & \xrightarrow{l} & \Gamma_j(\dot{T}M_{(0)}). \end{array}$$

**Remark 3.8** *Since the theorems of Church in [Chu12] and Randal-Williams [RW11] apply equally well to the component  $C_{\chi(M)/2}$ , one can rephrase the results of this section as follows. For any  $k \in \mathbb{Z}$  and an orientable  $n$ -manifold  $M$  of even Euler characteristic, the groups  $H_i(\text{Map}_0(M, S^n); \mathbb{Q})$  are isomorphic to  $H_i(\text{Map}_k(M, S^n); \mathbb{Q})$  for  $i \leq \chi(M)/2$ .*

## 4 Torsion

In this section, we describe how to modify the arguments of the previous section to compare the torsion in the homology of components of spaces of sections or configuration spaces. First we consider the case when  $n$  is odd. Then we describe when the  $p$ -torsion of  $\text{Map}_k(M, S^n)$  is isomorphic to the  $p$ -torsion of  $\text{Map}_j(M, S^n)$  and give a method for comparing the  $p$ -torsion in the homology of spaces of maps and spaces of sections. Finally, we draw new conclusions about stability for torsion in the homology of configuration spaces of particles in closed manifolds. Let  $\mathbb{Z}_{(p)}$  denote the  $p$ -local integers,  $S_{(p)}^n$  the  $p$ -local  $n$ -sphere and  $\dot{T}M_{(p)}$  the fiberwise  $p$ -localization.

**Proposition 4.1** *If  $n$  and  $p$  are odd, the connected components of  $\Gamma(\dot{T}M_{(p)})$  are all homotopic.*

**Proof** Since  $S_{(p)}^n$  is an  $H$ -space ([Sul05] page 43),  $\Gamma(\dot{T}M_{(0)})$  is also an  $H$ -space and hence its components are all homotopic.

The proof of Lemma 3.2 works with minimal modification to show the following proposition.

**Proposition 4.2** *If  $k/j$  is a unit in  $\mathbb{Z}_{(p)}$ , then the  $p$ -torsion of the groups  $H_i(\text{Map}_k(M, S^n); \mathbb{Z})$  and  $H_i(\text{Map}_j(M, S^n); \mathbb{Z})$  are isomorphic.*

The above proposition immediately applies to configuration spaces of particles in parallelizable manifolds. To use it to study non-parallelizable manifolds, we need the following generalization of Lemma 3.3, Proposition 3.4 and Proposition 3.5.

**Proposition 4.3** *If  $n$  is even and  $p \geq n/2 + 2$  or  $n = 2$ , and  $(2k - \chi(M))/(2j - \chi(M))$  is a unit in  $\mathbb{Z}_{(p)}$ , then there is a homotopy equivalence  $\Gamma_k(\dot{T}M_{(p)}) \simeq \Gamma_j(\dot{T}M_{(p)})$ .*

**Proof** If one knew that  $\text{Map}_d(S_{(p)}^n, S_{(p)}^n)$  were  $n+1$ -connected for all nonzero  $d$ , the proofs of Lemma 3.3, Proposition 3.4 and Proposition 3.5 would also work in this case. Since the space  $\text{Map}_d(S_{(p)}^n, S_{(p)}^n)$  is rationally  $n+1$ -connected, all we need to do is check that there is no torsion in the first  $n$  homotopy groups. Considering the following fibration:

$$\Omega_d^n S^n \longrightarrow \text{Map}_d(S^n, S^n) \longrightarrow S^n.$$

By the work of Serre in [Ser51], there is no  $p$ -torsion in  $\pi_i(\Omega_d^n S^n)$  for  $i \leq n$  since  $n = 2$  or  $p \geq n/2 + 2$ . Thus, the associated long exact sequence of homotopy groups show that  $\text{Map}_d(S_{(p)}^n, S_{(p)}^n)$  is  $n+1$ -connected.

Using Corollary 2.2, Proposition 4.1, Proposition 4.2 and Proposition 4.3, we get the following theorem.

**Theorem 4.4** *Let  $i \leq \min(k/2, j/2)$ . Then the  $p$ -torsion of  $H_i(C_k(M))$  and  $H_i(C_j(M))$  are isomorphic if at least one of the following four conditions are met:*

- 1)  $M$  is parallelizable and  $k/j$  is a unit in  $\mathbb{Z}_{(p)}$
- 2)  $n$  and  $p$  are both odd
- 3)  $n$  is even,  $p \geq n/2 + 2$  and  $(2k - \chi(M))/(2j - \chi(M))$  is a unit in  $\mathbb{Z}_{(p)}$
- 4)  $n = 2$  and  $(2k - \chi(M))/(2j - \chi(M))$  is a unit in  $\mathbb{Z}_{(p)}$ .

For example, part 1 of the above theorem implies that the 2-torsion of the homology of  $C_{2k+1}(M)$  stabilizes for  $M$  parallelizable. We can also apply part 4 of the theorem to study torsion in the homology of surface braid groups. For example, the previous theorem implies that the 2-torsion in the group homology of  $Br_{2k}(S^2)$  stabilizes where  $Br_k(S^2)$  is the spherical braid group on  $k$  strands. This (and the analogous statements at other primes) agrees with Fadell and Van Buskirk's calculation that  $H_1(Br_k(S^2)) = \mathbb{Z}/(2k-2)\mathbb{Z}$  (Theorem 2 of [FVB62]).

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